4.3 Counting Methods

4.3.1 What Are Counting Methods?

Counting methods are mathematical techniques used to determine how many ways certain events can happen. On the GMAT, these methods often appear in problems involving arrangements, selections, or combinations of items.

Think of it this way: if you're choosing outfits, arranging books on a shelf, or picking team members, you're solving a counting problem!

4.3.2 Multiplication Principle: The Foundation of Counting

The **multiplication principle** is the foundation of most counting problems. It states:

If one event can occur in m ways and another independent event can occur in n ways, then the total number of ways both events can occur together is:

$$\textbf{Total Outcomes} = m \times n$$

This principle extends to more than two events:

$$\textbf{Total Outcomes} = m_1 \times m_2 \times m_3 \times \cdots \times m_k$$

where each m_i represents the number of ways the i-th event can occur.

Example 1
Question: *You're picking an outfit. You have 3 shirts and 4 pairs of pants. How many unique outfits can you make?*

Solution:
This is a straightforward application of the multiplication principle. You have:

- *3 choices for shirts*
- *4 choices for pants*

Since these choices are independent, multiply the two:

$$\text{Total outfits} = 3 \times 4 = 12$$

So, there are 12 unique outfits possible.

4.3.3 Factorials: The Cornerstone of Counting

Factorials are at the heart of many counting formulas. A **factorial**, written as *n!*, is the product of all positive integers from 1 to *n*:

$$n! = n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1$$

By definition:

$$0! = 1$$

This special case is defined for consistency in formulas, especially combinations.

Example 2

- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $3! = 3 \times 2 \times 1 = 6$
- $1! = 1$

4.3.4 Combinations: When Order Doesn't Matter

When the order doesn't matter, you use combinations. This is about selecting groups, not arranging them.

The number of combinations of *r* items from a group of *n* is:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This expands to:

$$C(n, r) = \frac{n \times (n-1) \times \ldots \times (n-r+1)}{r \times (r-1) \times \ldots \times 1}$$

You're multiplying the first *r* descending numbers from *n*, and dividing by all the ways to rearrange those *r* items, because order doesn't matter.

Example 3

Question: *You're selecting 3 team members from a group of 6. How many different teams can you form?*

Solution:

$$C(6, 3) = \frac{6!}{3! \times (6-3)!} = \frac{6!}{3! \times 3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

GMAT Tip 1
Symmetry Identity
A key property of combinations is:

$$C(n, r) = C(n, n - r)$$

Choosing r items from n is the same as choosing the n - r items to leave out.

Why this is true:
Choosing r items to include is the same as choosing the n - r items to exclude. For example, choosing 3 people from a group of 10 is the same as choosing 7 people to leave out. Both lead to the same number of combinations.

4.3.5 Permutations: When Order Matters

When the order of items matters, you're dealing with permutations.
If you are selecting and arranging *r* items from a group of *n*, the number of **permutations** is given by:

$$P(n, r) = \frac{n!}{(n - r)!}$$

This expands to:

$$P(n, r) = n \times (n - 1) \times (n - 2) \times \ldots \times (n - r + 1)$$

You're multiplying the first *r* descending integers starting from *n*. Since order matters, every distinct arrangement is counted.

Example 4
Question: *You have 5 books and want to arrange 3 of them on a shelf. How many different arrangements are possible?*

Solution:

$$P(5, 3) = \frac{5!}{(5 - 3)!} = \frac{5!}{2!} = \frac{120}{2} = 60$$

⋆ GMAT Tip 2
A ***key property of permutations*** *is:*

$$P(n, r) = C(n, r) \times r!$$

Why this is true:
First, you use combinations to select r items from n:

$C(n, r)$ *ways to choose the group*

Then, you arrange those r items:

$r!$ *ways to order them*

So, the total number of ordered arrangements is:

$$P(n, r) = (\text{ways to choose}) \times (\text{ways to arrange}) = C(n, r) \times r!$$

4.3.6 ⋆ Common Counting Problem Types and Solving Strategies

GMAT counting questions often follow familiar patterns. You might be asked to arrange a set of items with repetition, form a group under certain conditions, or count outcomes with restrictions. For example: How many ways can a committee be selected from a mixed group of men and women? Or, how many seating arrangements are possible if two people insist on sitting together?

This section introduces the most common types of counting problems you'll encounter—along with efficient strategies to solve them quickly and accurately.

4.3.6.1 ⋆ Repeated Element Arrangement Problems

When some items in a list are identical, standard permutation formulas will overcount. To correct for this, divide by the factorial of each repeated item's count.

The number of distinct arrangements is given by the formula:

$$\frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$$

where:

- n is the total number of items
- $n_1, n_2, \ldots, n_k$ are the counts of each repeated item

⋆ Example 5

Question: *How many distinct ways can you arrange the letters in the word STATISTICS?*

Solution:

- *Total letters: 10*
- *Repetitions: 3 S's, 3 T's, 2 I's*

$$\frac{10!}{3! \times 3! \times 2!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3! \times 3! \times 2!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{3! \times 2!}$$

$$= 50{,}400$$

There are 50,400 distinct arrangements.

4.3.6.2 ⋆ Selection Problems with "At Least" or "At Most" Conditions

Some GMAT questions ask you to select a group that meets a condition such as **"at least"** or **"at most"** a certain number of members from a subgroup (e.g., at least 2 women on a team). These problems are often best solved using complementary counting—by subtracting the unwanted outcomes from the total.

Steps

When a question includes an "at least" or "at most" condition, it's often faster to work backward:

- Calculate the total number of possible outcomes (ignoring restrictions).
- Subtract the number of outcomes that do not meet the condition.

⋆ Example 6

Question: *A team of 5 people is selected from a group of 4 men and 4 women. How many ways can you form a team that includes at least 3 women?*

Solution:

Step 1: Calculate the total number of unrestricted teams

Choose 5 people from 8:

$$C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8!}{5! \times 3!} = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} = \frac{8 \times 7}{1} = 56$$

Step 2: Subtract the number of teams with fewer than 3 women

This includes two cases:

- *1 woman and 4 men*
- *2 women and 3 men*

Case 1: 1 woman and 4 men

$$C(4,1) \times C(4,4) = \frac{4!}{1! \times 3!} \times \frac{4!}{4! \times 1} = 4 \times 1 = 4$$

Case 2: 2 women and 3 men

$$C(4,2) \times C(4,3) = \frac{4!}{2! \times 2!} \times \frac{4!}{3! \times 1!} = 6 \times 4 = 24$$

Total teams with fewer than 3 women:

$$4 + 24 = 28$$

Step 3: Subtract from the total

$$56 - 28 = 28$$

There are 28 ways to form a team with at least 3 women.

⋆ Example 7

Question: *You roll a standard six-sided die 3 times. How many outcomes result in at least one 6?*

4.3.7 Summary of Key Rules and Formulas

1) Multiplication Principle

If one event can occur in m ways and another in n ways, the total number of ways both events can occur is:

$$\textbf{Total Outcomes} = m \times n$$

For multiple independent events:

$$\textbf{Total Outcomes} = m_1 \times m_2 \times m_3 \times \ldots \times m_k$$

where each m_i represents the number of ways the ith event can occur.

2) Factorials

The factorial of n, denoted as $n!$, is the product of all positive integers from 1 to n:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Special case:

$$0! = 1$$

3) Combinations

The number of ways to choose r items from a set of n items (Order doesn't matter):

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n \times (n-1) \times \ldots \times (n-r+1)}{r \times (r-1) \times \ldots \times 1}$$

Key property:

$$C(n, r) = C(n, n - r)$$

4) Permutations

The number of ways to arrange r items from a set of n items (Order matters):

$$P(n, r) = \frac{n!}{(n-r)!} = n \times (n - 1) \times (n - 2) \times \ldots \times (n - r + 1)$$

⋆ Key property:

$$P(n, r) = C(n, r) \times r!$$

5) ⋆ Repeated Element Arrangement

Number of distinct arrangements of n items with some repeated items is given by the formula:

$$\frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$$

4.3.8 ⋆GMAT-Style Questions

1) A restaurant offers 4 types of appetizers, 5 types of main courses, and 3 types of desserts. If a customer selects one item from each category, how many different meal combinations are possible?

A) 12
B) 30
C) 60
D) 75
E) 120

2) How many different ways can 4 students be arranged in a row for a group photo?

A) 12
B) 24
C) 48
D) 64
E) 120

3) A committee of 3 members is to be chosen from a group of 6 people. How many different ways can this committee be formed?

A) 10
B) 15
C) 20
D) 30
E) 60

4) How many distinct ways can the letters in the word "PEPPER" be arranged?

A) 60
B) 120
C) 180
D) 240
E) 360

5) Five people are to be seated around a circular table. In how many distinct ways can they be arranged?

A) 24
B) 48
C) 60
D) 120
E) 240

6) A company is assigning 6 projects to 3 consultants, with each

4.3.9 ⋆Answers

1) Topic: Multiplication Principle
Answer: C) 60

Explanation:
Use the Multiplication Principle, the total number of possible meal combinations is:

$$4 \times 5 \times 3 = 60$$

Thus, the correct answer is C) 60.

2) Topic: Permutations
Answer: B) 24

Explanation:
Since the order matters, we use the permutation formula:

$$P(n, n) = n!$$

For 4 students:

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Thus, the correct answer is B) 24.

3) Topic: Combinations
Answer: C) 20

Explanation:
Since the order doesn't matter, we use the combination formula:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

For this problem:

$$C(6, 3) = \frac{6!}{3!(6-3)!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

Thus, the correct answer is C) 20.

4) Topic: Repeated Element Arrangement
Answer: A) 60

Explanation:
The word "PEPPER" has 6 letters, with repeated letters:

- *3 P's*
- *2 E's*
- *1 R's*